VARIANCE OF LIPSCHITZ FUNCTIONS AND AN ISOPERIMETRIC PROBLEM FOR A CLASS OF PRODUCT MEASURES*

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July 10, 1995

Abstract

The maximal variance of Lipschitz functions (with respect to the ℓ_1–distance) of independent random vectors is found. This is then used to solve the isoperimetric problem, uniformly in the class of product probability measures with given variance.

1 Statements

Let $\xi = (\xi_1, \cdots, \xi_n)$ be a vector of independent random variables with finite variance $\sigma_i^2 = \mathbf{Var}\, \xi_i$, $1 \leq i \leq n$. Denote by $\mathcal{F}_1$ the class of all functions on $\mathbf{R}^n$ which are Lipschitz with respect to the ℓ^1–distance

$$d_1(x,y) = \|x-y\|_1 = \sum_{k=1}^{n} |x_k - y_k|, \quad x, y \in \mathbf{R}^n.$$

By definition, $f \in \mathcal{F}_1$, if for all $x, y \in \mathbf{R}^n$, $|f(x) - f(y)| \leq d_1(x,y)$. Let $S_n = \xi_1 + \ldots + \xi_n$.

*Key Words: Isoperimetry, Lipschitz function, Variance inequality
†Research supported in part by the ISF grant NZX000 and NZX300
‡Research supported in part by an NSF Mathematical Sciences Postdoctoral Fellowship

Theorem 1 *In the class $\mathcal{F}_1$, the maximal value of* $\mathbf{Var} f(\xi)$ *is attained at the function $f(x) = x_1 + \ldots + x_n$. In other words, for any $f \in \mathcal{F}_1$,*

$$\mathbf{Var} f(\xi) \ \leq \ \mathbf{Var} S_n = \sum_{i=1}^n \sigma_i^2. \tag{1.1}$$

X.Fernique [F, Théorème 3.2] proved an inequality similar to (1.1) for $f \in \mathcal{F}_1$, convex. However, ξ there, is only assumed to be symmetrically distributed, i.e., for all $\varepsilon_i = \pm 1$, the random vectors $(\varepsilon_1\xi_1, \cdots, \varepsilon_n\xi_n)$ have the same distribution (of course, this is fulfilled if the ξ_i are i.i.d. with a symmetric one–dimensional distribution). In contrast to the difficult proof of Fernique, Theorem 1 can easily be obtained by induction.

Theorem 1 also has the following consequence: Denote by $M^n(\sigma)$ the family of all the product measures $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ on $\mathbf{R}^n$ with given variance $\mathbf{Var}\,(\mu) = \sigma^2$, where

$$\mathbf{Var}\,(\mu) = \sum_{i=1}^n \int_{\mathbf{R}} \left| x - \int_{\mathbf{R}} t d\mu_i(t) \right|^2 d\mu_i(x).$$

Hence, with the above notations, $\mathbf{Var}(\mu) = \mathbf{Var} S_n$. Now, given a set $A \subset \mathbf{R}^n$ and $h > 0$, denote by

$$A^h = A + hB_1 = \{x \in \mathbf{R}^n : d_1(a, x) < h, \text{ for some } a \in A\}$$

the open h–neighbourhood of A (B_1 is the open ℓ^1–unit ball in $\mathbf{R}^n$). From Theorem 1 we obtain a solution to the isoperimetric problem with respect to the ℓ^1–distance uniformly in the class $M^n(\sigma)$ controlled by the parameter σ.

Theorem 2 *For any $h > 0$, $\sigma > 0$ and $p \in (0, 1)$,*

$$\inf_{\mu \in M^n(\sigma)} \ \inf_{\mu(A) \geq p} \mu(A^h) \ = \ \begin{cases} p, & \text{if} \quad h \leq \frac{\sigma}{\sqrt{p(1-p)}}, \\ 1 - \frac{p\sigma^2}{ph^2 - \sigma^2}, & \text{if} \quad h \geq \frac{\sigma}{\sqrt{p(1-p)}}. \end{cases} \tag{1.2}$$

The first infimum in (1.2) is taken over all the $\mu \in M^n(\sigma)$, the second is taken over all the Borel sets A of μ–measure greater or equal to p. In particular, from Theorem 2, we have:

Corollary 3 *Given $\sigma > 0$ and $p \in (0,1)$, one can garantee that $\mu(A^h) > p$ regardless of the dimension $n \geq 1$, regardless of the measure $\mu \in M^n(\sigma)$, and regardless of the set $A \subset \mathbf{R}^n$ of μ–measure greater or equal to p, if and only if*

$$h > h(p,\sigma) \equiv \frac{\sigma}{\sqrt{p(1-p)}}.$$

Otherwise, it is possible to have $\mu(A^h) = p$.

Equality in (1.2) is already attained when $n = 1$. Indeed, denote by δ_x the unit mass at the point $x \in \mathbf{R}$. If $h \leq h(p,\sigma)$, take

$$\mu = p\delta_0 + (1-p)\delta_{h(p,\sigma)}, \quad A = \{0\}.$$

Then, $\mathbf{Var}\,(\mu) = \sigma^2$, $A^h = (-h,h)$, so $\mu(A^h) = p = \mu(A)$. If $h \geq h(p,\sigma)$, take

$$\mu = p\delta_0 + q\delta_x + r\delta_h, \quad A = \{0\},$$

with $r = p\sigma^2/(ph^2 - \sigma^2)$, $q = 1 - p - r$, $x = rh/(p+q)$. Then, it is again easy to verify that $\mathbf{Var}\,(\mu) = \sigma^2$, and that $\mu(A^h) = 1 - p\sigma^2/(ph^2 - \sigma^2)$.
Since equality in (1.2) is attained when $n = 1$, (1.2) will not change if the h–neighbourhood is defined with respect to the ℓ^2–distance, or more generally, with respect to the ℓ^α–distance in $\mathbf{R}^n$, $1 \leq \alpha \leq +\infty$. Indeed, the ℓ^α–unit ball B_α is larger than B_1, hence, $A + hB_1 \subset A + hB_\alpha$, and therefore $\mu(A + hB_1) \leq \mu(A + hB_\alpha)$. Hence, the same inequality holds when one takes the second infimum in (1.2). But, all the balls B_α coincide when $n = 1$ (in which case equality in (1.2) is attained).

For individual measures μ (for example, for those having finite exponential moments) there exist estimates for $1 - \mu(A^h)$ which decrease exponentially when $h \to +\infty$ (see M.Talagrand [T2]). For example, given $\sigma_i > 0$, $1 \leq i \leq n$, let $\mu = \mu_1 \otimes \cdots \otimes \mu_n \in M^n(\sigma)$, where $\mu_i = (\delta_{\sigma_i} + \delta_{-\sigma_i})/2$, $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$. Then, as shown in [T2, Proposition 2.1.1, Theorem 2.4.1] (see also M.Ledoux [L, p.24] for an extension to non–identical marginals), if $h > 0$, $\mu(A) = p$, then

$$\mu(A + hB_1) \geq 1 - \frac{1}{p}\exp(-h^2/4\sigma^2). \tag{1.3}$$

When all the $\sigma_i = 1$, the extremal sets minimizing $\mu(A^h)$, while $\mu(A) = p$ is fixed, are known and were obtained by L.H.Harper [H]. If one minimizes

$\mu(A^h)$ over all convex sets A, the situation changes considerably, and we then deal with a much more powerful concentration principle discovered by M.Talagrand (see [T1],[T2]). In particular, when all the $\sigma_i = 1$, one has

$$\mu(A + hB_2) \geq 1 - \frac{1}{p}\exp(-h^2/8).$$

In our case, since one looks for a uniformly minimal value of $\mu(A + hB_1)$, it does not matter whether one considers convex sets or arbitrary sets, since the extremal $A = \{0\}$ is convex.

To finish this section, we give an inequality which is actually equivalent to the second part of (1.2). For non–empty sets $A, B \subset \mathbf{R}^n$, let $d_1(A, B) = \inf\{d_1(a, b) : a \in A,\, b \in B\}$.

Corollary 4 *For any $\mu \in M^n(\sigma)$, and any non–empty Borel sets $A, B \subset \mathbf{R}^n$,*

$$d_1(A, B) \leq \sigma\sqrt{\frac{1}{\mu(A)} + \frac{1}{\mu(B)}}. \tag{1.4}$$

Let $\mu(A) > 0$, $\mu(B) > 0$ be such that $\mu(A) + \mu(B) \leq 1$. Then, choosing $B = \{h\}$ with h equal to the right–hand side of (1.4), it is easily seen that equality in (1.4) is attained at the same measure μ and the same set $A = \{0\}$ as the second inequality in (1.2).

2 Proofs

A statement slightly more general than Theorem 1 will actually be proved. Assume we have n measurable spaces (X_k, Σ_k) and n measurable functions $h_k = h_k(x_k, y_k)$ defined on $X_k \times X_k$, $1 \leq k \leq n$, and which vanish on the diagonal $x_k = y_k$. Let ξ_k be independent random variables with values in X_k, $1 \leq k \leq n$, such that

$$2\sigma_k^2 = \mathbf{E}h_k^2(\xi_k, \eta_k) < +\infty,$$

where η_k is an independent copy of ξ_k. Put $\xi = (\xi_1, \cdots, \xi_n)$.

Lemma 5 *Let f be a measurable function defined on $X_1 \times \ldots \times X_n$ such that*

$$|f(x) - f(y)| \;\leq\; \sum_{k=1}^{n} h_k(x_k, y_k), \tag{2.1}$$

for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X_1 \times \ldots \times X_n$. *Then,*

$$\mathbf{Var} f(\xi) \;\leq\; \sum_{k=1}^{n} \sigma_k^2. \tag{2.2}$$

Proof. This lemma is proved by induction on the dimension n. For $n = 1$, and since $2\mathbf{Var} f(\xi) = \int\int (f(\xi) - f(\eta))^2 d\mu(\xi) d\mu(\eta)$, (2.2) is immediate. Assume now that (2.2) is true for n. Denote by μ_{n+1} the distribution of ξ_{n+1}, and by P_n the distribution of the random vector $(\xi_1, \cdots, \xi_n)$, thus $P_{n+1} = P_n \otimes \mu_{n+1}$ is the distribution of $(\xi_1, \cdots, \xi_{n+1})$. Let $f : X_1 \times \ldots \times X_{n+1} \longrightarrow \mathbf{R}$ satisfy (2.1). Now, fix x_{n+1}. Since the function $g(x_1, \cdots, x_n) = f(x_1, \cdots, x_n, x_{n+1})$ satifies (2.1), making use of the induction hypotheses and writing (2.2) for g we get:

$$\int g^2 dP_n \;\leq\; \left(\int g dP_n\right)^2 + \sum_{k=1}^{n} \sigma_k^2. \tag{2.3}$$

The function $m(x_{n+1}) = \int g dP_n$ is well–defined, measurable and as a function of one variable,

$$|m(x_{n+1}) - m(y_{n+1})| \leq h_{n+1}(x_{n+1}, y_{n+1}).$$

Thus m satisfies (2.1), hence

$$\int m^2 d\mu_{n+1} \;\leq\; \left(\int m d\mu_{n+1}\right)^2 + \sigma_{n+1}^2. \tag{2.4}$$

Integrating (2.3) over X_{n+1} (with respect to μ_{n+1}), and taking into account (2.4), gives (2.2) for f. Lemma 5 and thus Theorem 1 are proved.

Proof of Theorem 2

Let $A \subset \mathbf{R}^n$ be such that $\mu(A) \geq p$. Since the function $f(x) = \inf_{a \in A} d_1(a, x)$ belongs to $\mathcal{F}_1$, we have by Theorem 1, that $\mathbf{Var} f \leq \sigma^2$. In addition, $f \geq 0$ and $\mu(f = 0) \geq p$. Note also that $A^h = \{x \in \mathbf{R}^n : f(x) < h\}$. To get (1.2), it just remains to appeal to the following result:

Lemma 6 *For any* $h > 0$, $\sigma > 0$ *and* $p \in (0, 1)$,

$$\sup\; \mathbf{P}(\xi \geq h) \;=\; \begin{cases} 1 - p, & \textit{if} \quad h \leq \frac{\sigma}{\sqrt{p(1-p)}}, \\ \frac{p\sigma^2}{ph^2 - \sigma^2,} & \textit{if} \quad h \geq \frac{\sigma}{\sqrt{p(1-p)}}, \end{cases} \tag{2.5}$$

where the supremum is taken over all non–negative random variables ξ on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$ such that $\mathbf{P}(\xi = 0) \geq p$ and $\mathbf{Var}\,\xi \leq \sigma^2$.

Proof. Denote by $\mathcal{L}(\xi)$ the distribution of ξ. The cases of equality in (2.5) were, in fact, already settled in Section 1. If

$$h \leq h(p,\sigma), \quad \mathcal{L}(\xi) = p\delta_0 + (1-p)\delta_{h(p,\sigma)},$$

then $\mathbf{Var}\,(\xi) = \sigma^2$, $\mathbf{P}(\xi = 0) = p$, $\mathbf{P}(\xi \geq h) = 1 - p$. If

$$h > h(p,\sigma), \quad \mathcal{L}(\xi) = p\delta_0 + q\delta_x + r\delta_h,$$

where $r = p\sigma^2/(ph^2 - \sigma^2)$, $q = 1-p-r$, $x = rh/(p+q)$, then as easily verified, we have $\mathbf{Var}\,(\xi) = \sigma^2$, $\mathbf{P}(\xi = 0) = p$, and $\mathbf{P}(\xi \geq h) = r = p\sigma^2(ph^2 - \sigma^2)$. So, one needs only to show that whenever $h \geq h(p,\sigma)$,

$$\mathbf{P}(\xi \geq h) \leq \frac{p\sigma^2}{ph^2 - \sigma^2}. \tag{2.6}$$

To prove this last statement, several steps will be needed.

Step 1 (reduction to the bounded case).
It is easy to see, via a usual truncation argument, that ξ can be assumed to be bounded, say, by a constant C. So, $\mathcal{L}(\xi)$ is assumed to be a probability measure F on $[0, C]$.

Step 2 (reduction to discrete case with only four atoms).
Fixing

$$p_0 = \mathbf{P}(\xi = 0) = F(\{0\}), \ a = \mathbf{E}\xi = \int_0^C x dF(x), \ b = \mathbf{E}\xi^2 = \int_0^C x^2 dF(x),$$

defines a subset $Z(p_0, a, b)$ of the family Z of all probability measures on $[0, C]$. Then, observe the following:

1) Z is a (convex, compact for the topology of weak convergence) symplex in the space M of all signed measures on $[0, C]$, and the extremal points of Z are just the unit masses δ_x, $0 \leq x \leq C$.

2) $Z(p_0, a, b)$ is the intersection of Z with three hyperspaces $H_0 = \{F \in M : F(\{0\}) = p_0\}$, $H_1 = \{F \in M : \int_0^C x dF(x) = a\}$ and $H_2 = \{F \in M : \int_0^C x^2 dF(x) = b\}$.

3) Therefore, noting that H_1 and H_2 are closed, while $clos(H_0) = \{F \in M : F(\{0\}) \geq p_0\}$, one easily concludes that the extremal points of $Z(p_0, a, b)$ are just the measures of the form $F = p_0'\delta_0 + p_1\delta_{x_1} + p_2\delta_{x_2} + p_3\delta_{x_3}$, with $p_0' \geq p_0$.

4) The functional $F \longrightarrow F([h, +\infty))$ is linear and, since the set $[h, +\infty)$ is closed, this functional is upper semi–continuous on Z. Hence, its maximum on $Z(p_0, a, b)$ is attained at an extremal point of $Z(p_0, a, b)$.

Of course, the above maximum should be further maximized over all possible p_0, a, b (in particular, we should have $p_0 \geq p$, $a^2 \leq b$, $b - a^2 \leq \sigma^2$). But in the sequel, one needs only consider the ξ taking four values, and moreover, on may also forget about the condition $\xi \leq C$.

Step 3 (reduction to the four atoms case, one of them being h).
So, by the previous step, it suffices to prove (2.6) when

$$\mathcal{L}(\xi) = p_0\delta_0 + p_1\delta_{x_1} + p_2\delta_{x_2} + p_3\delta_{x_3},$$

$p_0 \geq p$, $p_1, p_2, p_3 \geq 0$, $p_0 + p_1 + p_2 + p_3 = 1$, $x_1, x_2, x_3 \geq 0$. Thus we need to maximize the functional

$$J = p_1\mathbf{1}_{\{x_1 \geq h\}} + p_2\mathbf{1}_{\{x_2 \geq h\}} + p_3\mathbf{1}_{\{x_3 \geq h\}}$$

under the condition

$$u(x_1, x_2, x_3) \equiv \mathbf{Var}\,\xi = (p_1x_1^2 + p_2x_2^2 + p_3x_3^2) - (p_1x_1 + p_2x_2 + p_3x_3)^2 \leq \sigma^2.$$

Let us keep p_0, p_1, p_2, p_3 fixed, fixed for a while. Next, one can assume that $0 \leq x_1 \leq x_2 \leq x_3$. Note that in the region $x_3 > x_2$,

$$\frac{\partial u}{\partial x_3} = 2p_3(x_3 - (p_1x_1 + p_2x_2 + p_3x_3)) > 0.$$

That is, $u = \mathbf{Var}\,\xi$ is an increasing function of x_3 in $[x_2, +\infty)$. But, as a function of x_3, J is constant in $[0, h)$ and constant $[h, +\infty)$. Hence, any $x_3 \in [0, h)$ can be replaced by $x_3 = x_2$, and any $x_3 \in [h, +\infty)$ can be replaced by $x_3 = \max(x_2, h)$. In the first case, a new random variable ξ will have a smaller variance, the value of J will be unchanged, but ξ will take at most three value: $0, x_1, x_2$. In the second case, the same is true provided that $x_2 \geq h$, and ξ will take the values $0, x_1, x_2, h$ if $x_2 < h$. Now, let $x_2 \geq h$, thus one can put $x_3 = x_2$. Then, by the same type of reasoning, x_2 can be replaced by $x_2 = \max(x_1, h)$. If $x_1 \geq h$, then ξ will take only the values 0 and x_1, and once more x_1 can be replaced by h.

The arguments above show that only the case $0 \leq x_1, x_2 < x_3 = h$ need to be worked out and in fact the value $x_3 = h$ may disapper if $p_3 = 0$

Step 4 (reduction to the three atoms case, one of them being h).
Proving (2.6) is thus reduced to study random variables ξ such that

$$\mathcal{L}(\xi) = p_0\delta_0 + p_1\delta_{x_1} + p_2\delta_{x_2} + p_3\delta_h,$$

where $p_0 \geq p,\ p_1, p_2, p_3 \geq 0,\ p_0 + p_1 + p_2 + p_3 = 1,\ 0 \leq x_1, x_2 < h$. Now, $J = p_3$ and one needs to maximize the functional J under the condition

$$u(x_1, x_2) \equiv \mathbf{Var}\,\xi = (p_1x_1^2 + p_2x_2^2 + p_3h^2) - (p_1x_1 + p_2x_2 + p_3h)^2 \leq \sigma^2.$$

Again, p_0, p_1, p_2, p_3 are kept fixed and x_1 and x_2 are allowed to vary in $[0, h)$. In addition, it is assumed that, for p_0, p_1, p_2, p_3 given and fixed, there exists at least one pair $x_1, x_2 \in [0, h)$ such that $u(x_1, x_2) \leq \sigma^2$ (otherwise, the above values of p_0, p_1, p_2, p_3 cannot be considered).

Under the above conditions, $J = p_3$ is constant, and there is nothing to maximize. However, one needs to prove the existence of p_0, p_1, p_2, p_3 such that $u(x_1, x_2) \leq \sigma^2$, for some $x_1, x_2 \in [0, h)$. To finish this step, we only show that if this is at all possible, one may choose $x_1 = x_2$.

First observe that $u = u(x_1, x_2)$ is a positive quadratic form, hence an extremal point of u will also also be the global minimum on the whole plane $\mathbf{R}^2$. The extremal point satisfies

$$\begin{cases} \frac{1}{2}\frac{\partial u}{\partial x_1} = p_1x_1 - p_1(p_1x_1 + p_2x_2 + p_3h) = 0 \\ \frac{1}{2}\frac{\partial u}{\partial x_2} = p_2x_2 - p_2(p_1x_1 + p_2x_2 + p_3h) = 0 \end{cases}$$

that is, assuming that $p_1, p_2 > 0$,

$$\begin{cases} x_1 = p_1x_1 + p_2x_2 + p_3h \\ x_2 = p_1x_1 + p_2x_2 + p_3h \end{cases}$$

from which it follows that

$$x_1 = x_2 = \frac{p_3}{1 - p_1 - p_2}h. \tag{2.7}$$

Now, since $p_3 < 1 - p_1 - p_2 = p_0 + p_3$, then $x_1 = x_2 \in [0, h)$. Therefore, the global minimum (x_1, x_2) of u on $\mathbf{R}^2$ is also a global minimum of u on the square $[0, h) \times [0, h)$. Therefore, there exists $(x_1, x_2) \in [0, h) \times [0, h)$ such that $u(x_1, x_2) \leq \sigma^2$, if and only if $u(x_1, x_2) \leq \sigma^2$ with $x_1 = x_2$ defined by (2.7). When, $p_1 = 0$ and/or $p_2 = 0$ we, in fact, deal with ξ taking at most three values.

Step 5: the case $F = p_0\delta_0 + p_1\delta_x + p_2\delta_h,\ \ 0 \leq x < h,\ \ p_0 \geq p,\ \ p_1, p_2 \geq 0$, $p_0 + p_1 + p_2 = 1$.

Again, $J = p_2$ is constant. According to (2.7), (changing the notations there), for fixed p_0, p_1, p_2, the minimal value of $\mathbf{Var}\,\xi$ as a function of x on $[0, h)$ is attained at

$$x = \frac{p_2}{p_0 + p_2} h.$$

For this value of x, we find

$$\begin{aligned}
\mathbf{Var}\,\xi &= (p_1x^2 + p_2h^2) - (p_1x + p_2h)^2 \\
&= \left(p_1\frac{p_2^2}{(p_0+p_2)^2}h^2 + p_2h^2\right) - \left(p_1\frac{p_2}{p_0+p_2}h + p_2h\right)^2 \\
&= \left[\frac{p_2}{(p_0+p_2)^2}(p_1p_2 + (p_0+p_2)^2) - \frac{p_2^2}{(p_0+p_2)^2}(p_1 + (p_0+p_2))^2\right] h^2 \\
&= \frac{p_2}{(p_0+p_2)^2}\left[p_1p_2 + (p_0+p_2)^2 - p_2\right] h^2 \\
&= \frac{p_0p_2}{p_0+p_2}h^2.
\end{aligned}$$

Now, we have to maximize $J = p_2$ under the condition

$$\mathbf{Var}\,\xi = \frac{p_0p_2}{p_0+p_2}h^2 \leq \sigma^2. \tag{2.8}$$

From (2.8), when p_0 decreases, $\mathbf{Var}\,\xi$ also decreases, while $J = p_2 = 1 - p_0 - p_1$ increases (p_1 is fixed). Hence, in the sequel, it is enough to only consider the case $p_0 = p$. The possible maximal value $p_2 = 1 - p$ satisfies (2.8) if and only if $p(1-p) \leq \sigma^2/h^2$, that is, if and only if $h \leq h(p, \sigma)$. Otherwise, if $h > h(p, \sigma)$, or even if $h = h(p, \sigma)$, the maximal value of $J = p_2$ is, according to (2.8), the value which satisfies

$$\frac{pp_2}{p + p_2}h^2 = \sigma^2.$$

The only solution to this equation is given by

$$p_2 = \frac{p\sigma^2}{ph^2 - \sigma^2}.$$

Lemma 6 follows.

Proof of Corollary 4.
Let $p = \mu(A)$, $q = \mu(B)$. If $p = 0$ or $q = 0$, there is nothing to prove. If $p+q > 1$, then $A \cap B \neq \emptyset$, so $d_1(A, B) = 0$. Thus, we need only consider the case $p + q \leq 1$. Let $p, q > 0$, $p + q \leq 1$, and assume $A \cap B = \emptyset$. Note that

$$h \equiv \sigma\sqrt{\frac{1}{p} + \frac{1}{q}} > h(p, \sigma).$$

Therefore, by (1.2),

$$1 - \mu(A^h) \leq \frac{p\sigma^2}{ph^2 - \sigma^2} = q,$$

and again by (1.2), for all $h_1 > h$, $1 - \mu(A^{h_1}) < q$. Hence, $B \cap (A^{h_1} \setminus A) \neq \emptyset$, and therefore, $d_1(A, B) \leq h_1$. Letting $h_1 \to h$, completes the proof.

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